An introduction to ICA

followed by:

EM Algorithms for ICA

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Parietal

Joint work with: F. Bach, JF. Cardoso & A. Gramfort

https://arxiv.org/abs/1805.10054

Parietal presentation, 2018

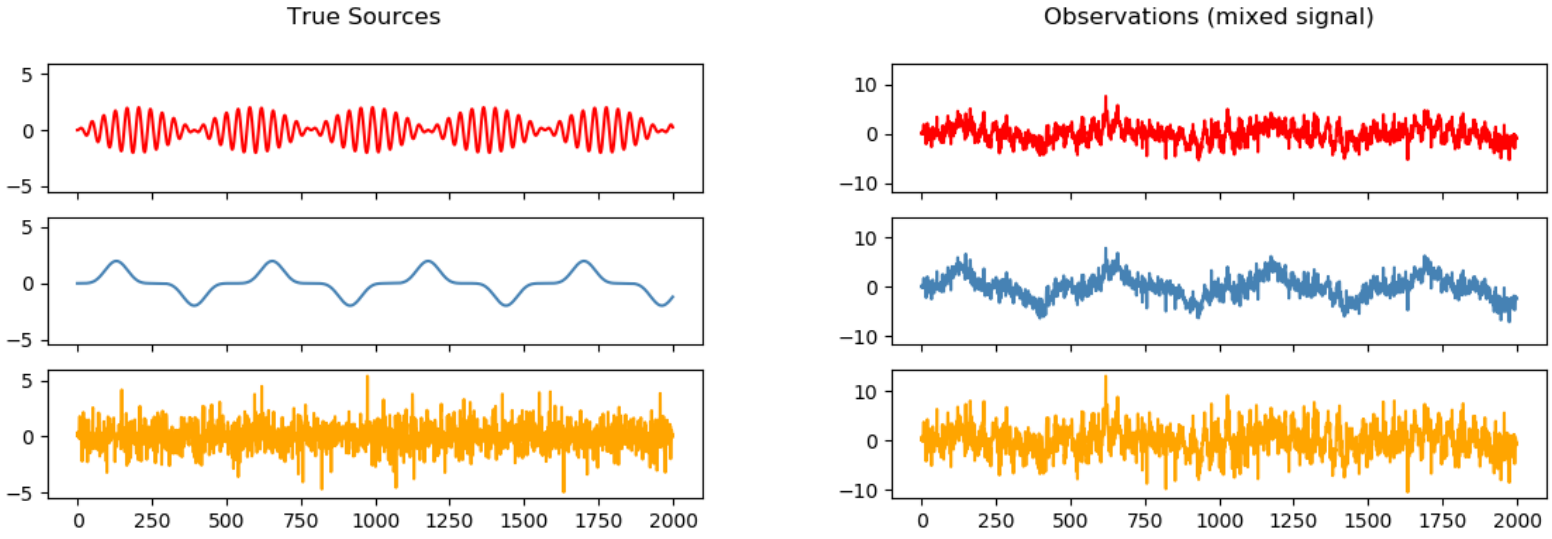
Introduction to ICA

Source separation: the cocktail party problem

Independent component analysis

Special case of source separation:

- Linear & instantaneous mixture
- "Square problem": as many sources as sensors



$$x_1 = 1.1s_1 + 0.9s_2 + 1.2s_3$$

$$x_2 = 0.5s_1 + 0.8s_2 + 2.2s_3$$

$$x_3 = 1.5s_1 + 0.5s_2 - 2.4s_3$$

Problem formulation: ICA as a generative model

- We observe p signals $[x_1, \cdots, x_p] = \mathbf{x} \in \mathbb{R}^{p \times 1}$

Key assumption

There are p independent signals $[s_1, \cdots, s_p] = \mathbf{s} \in \mathbb{R}^{p \times 1}$ and $A \in \mathbb{R}^{p \times p}$ invertible such that:

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Problem formulation

$$\mathbf{x} = A\mathbf{s}$$

Given some realizations of $\mathbf{x}$, we want to recover A and $\mathbf{s}$.

Is it possible?

Standard indeterminations:

- No hope to recover sources scales

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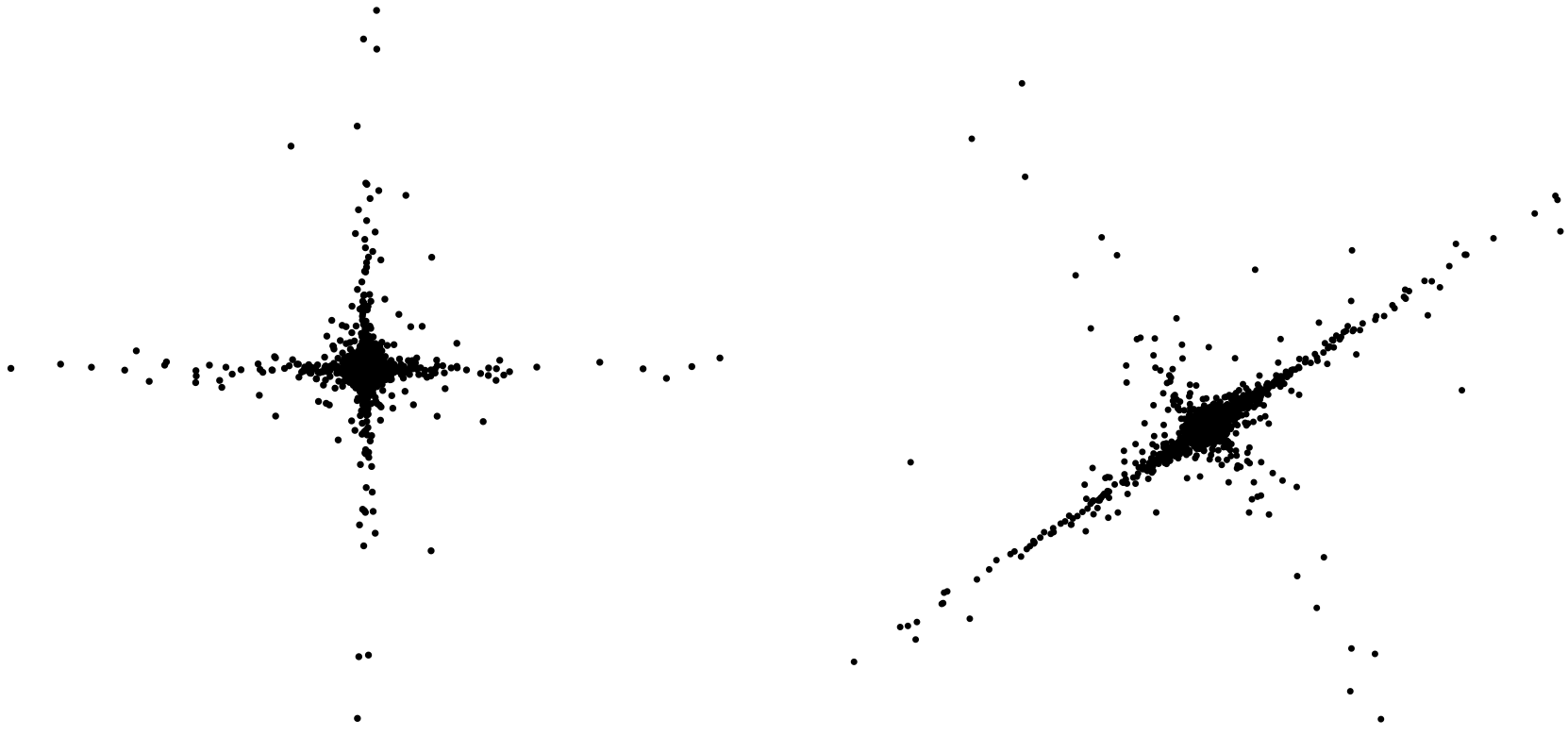
A geometric viewpoint

In 2D ($p = 2$). $n = 2000$ points.



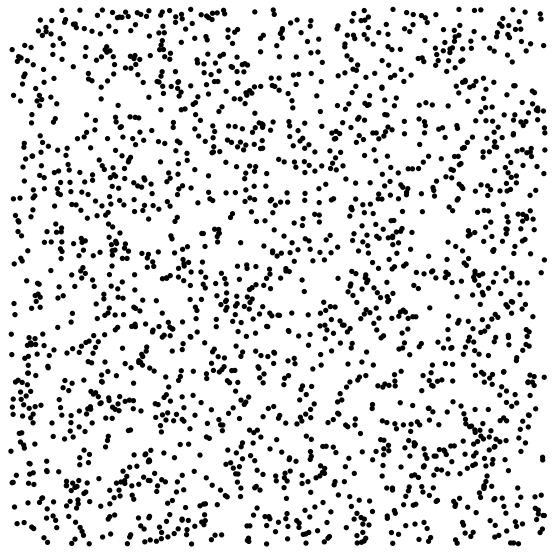
Sources

Mixed observed signals

Density matters

Different densities lead to different patterns

Super-Gaussian



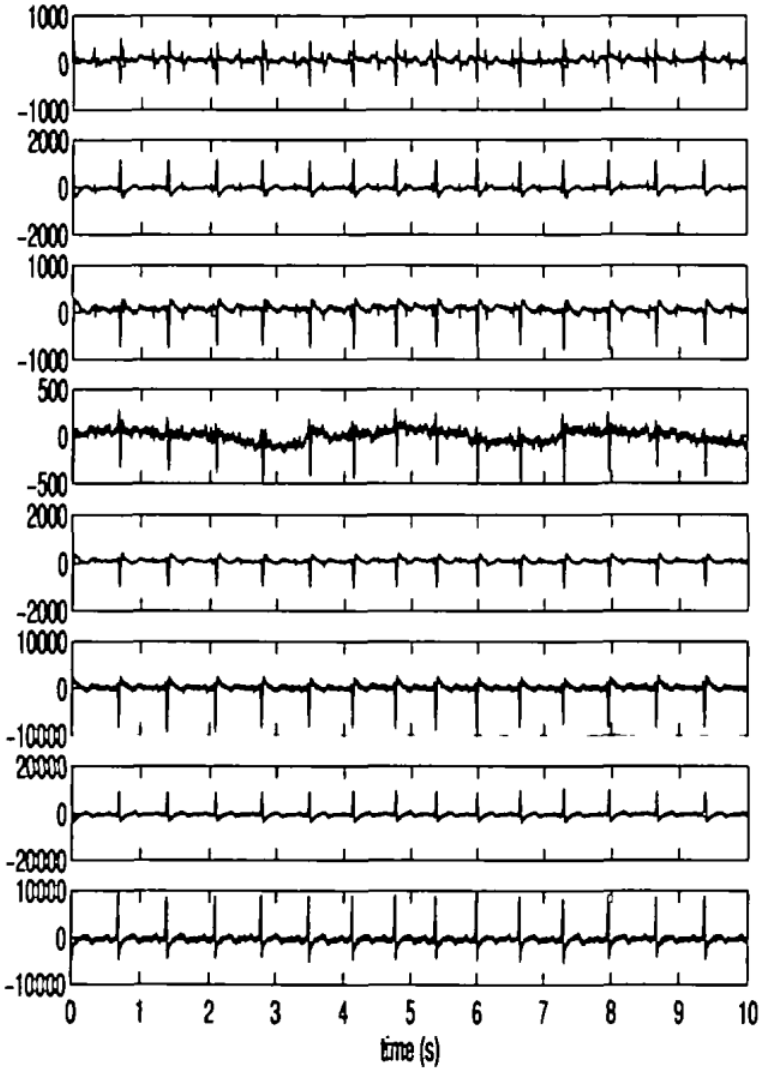
Sub-Gaussian

Gaussian

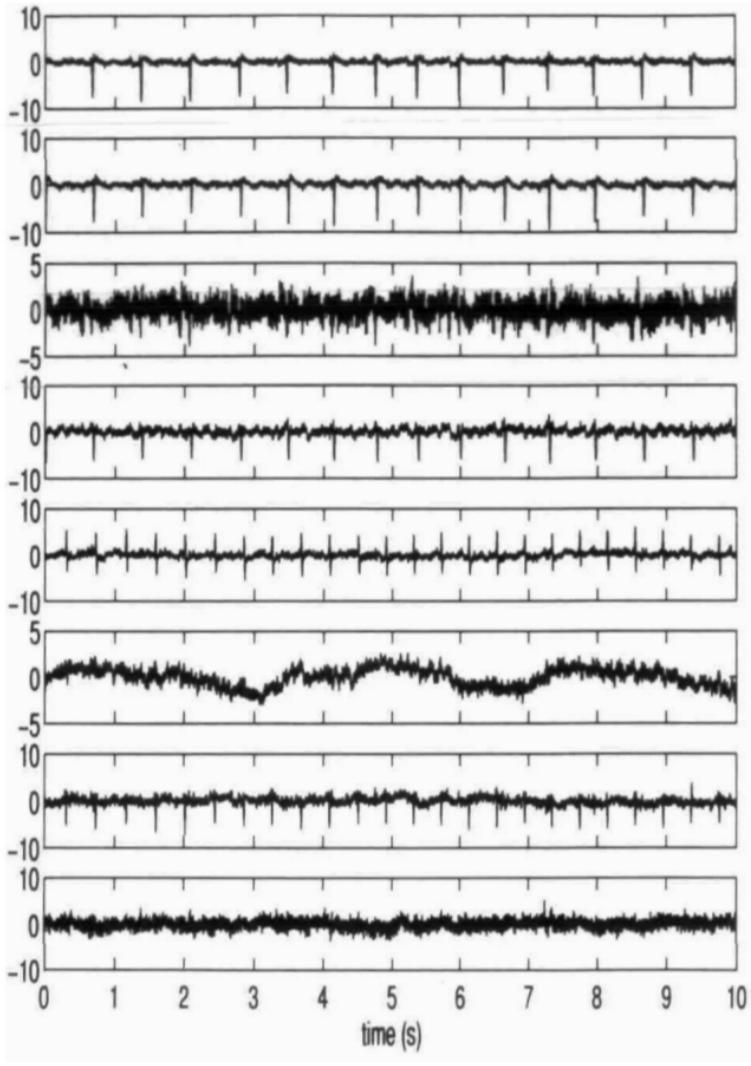
ICA in the real world

A cute example

ECG of a pregnant mother

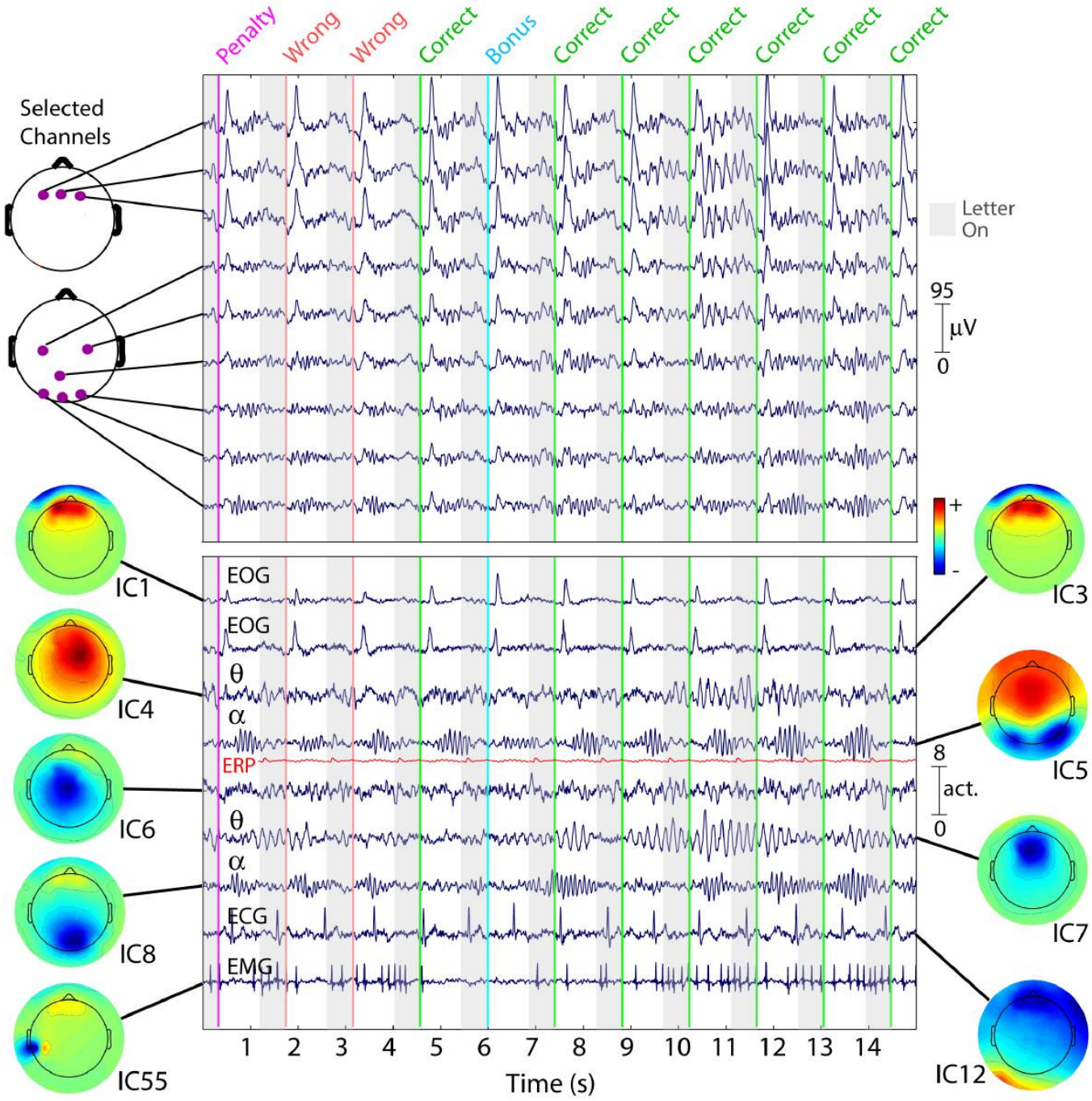


Recovered ICA sources

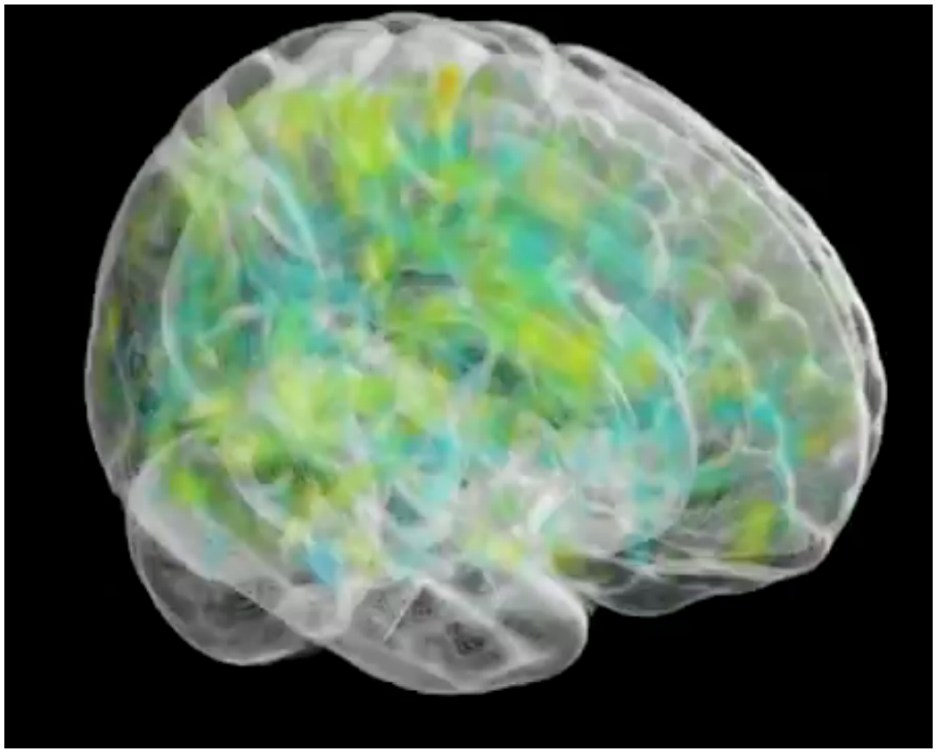


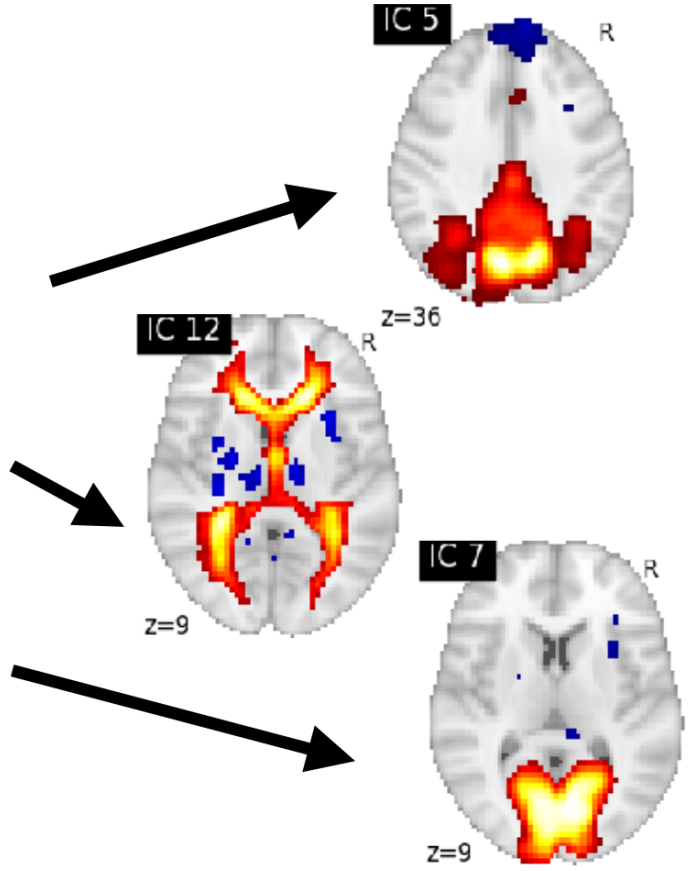
[Zarzoso '97]

ICA on EEG-MEG data



ICA on fMRI





A matrix factorization problem

Link with dictionary learning

Given n samples noted in matrix form $X \in \mathbb{R}^{p \times n}$

ICA: Find $A \in \mathbb{R}^{p \times p}$ and $S \in \mathbb{R}^{p \times n}$ such that $X = AS$.

- Perfect data fit ($X \boxed{=} AS$)
- Assumption of statistical independence on S

Dictionnary learning: Find $D \in \mathbb{R}^{p \times k}$ and $R \in \mathbb{R}^{k \times n}$ such that $X \simeq DR$

- Approximate data fit (introduces a penalty $||X - DR||_F$ in the optimization)
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Inference techniques

Maximum likelihood ICA

- $\mathbf{x} = A\mathbf{s}$: generative model.
- Further assumption: fixed density. $\boxed{s_i \sim d}$

Likelihood:

$$\boxed{p(\mathbf{x}|A) = \frac{1}{|\det(A)|} \prod_{i=1}^{p} d([A^{-1}\mathbf{x}]_i)}$$

Optimization problem

- Work with the unmixing matrix $W = A^{-1}$
- Cost function $\ell(\mathbf{x}, W) = -\log(p(\mathbf{x}|W^{-1}))$

$$\ell(\mathbf{x}, W) = -\log|\det(W)| - \sum_{i=1}^{p} \log(d([W\mathbf{x}]_i))$$

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Optimization problem 2

Objective of maximum-likelihood ICA: find

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Infomax

Stochatic gradient descent:

$$W_{t+1} = W_t - \rho \nabla \mathcal{L}_n(W_t)$$

The gradient is computed on a mini-batch of samples.

Issues

- Choosing ρ is critical and difficult (non-convex problem)
- No safe rule / descent guarantee
- Too small : slow convergence
- Too large : blow-up
- Line-search is hard in a stochastic setting

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Proposed method

- Stochastic, so fast
- Guaranteed descent at each iteration
- One iteration is as costly as SGD

EM algorithms for ICA

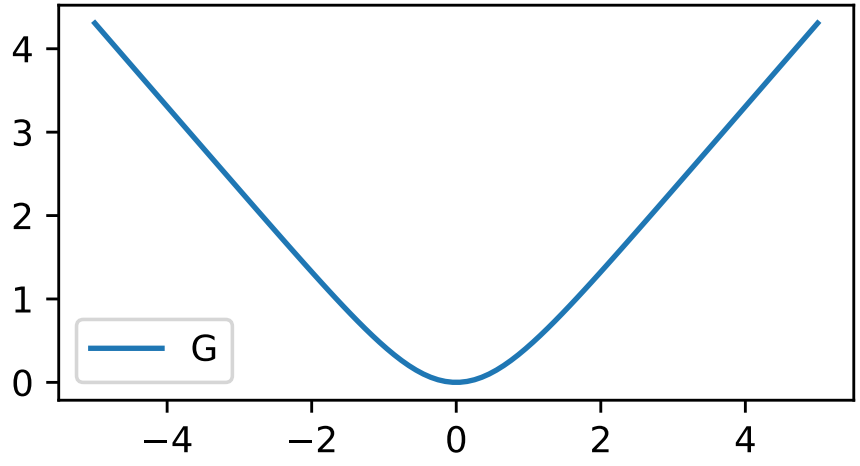
Super-Gaussian densities

- Define $G(y) = -\log(d(y))$.
- $\mathcal{L}_n(W) = -\log|\det(W)| + \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n} G([WX]_{ij})$

Key assumption : d is *super-Gaussian*.

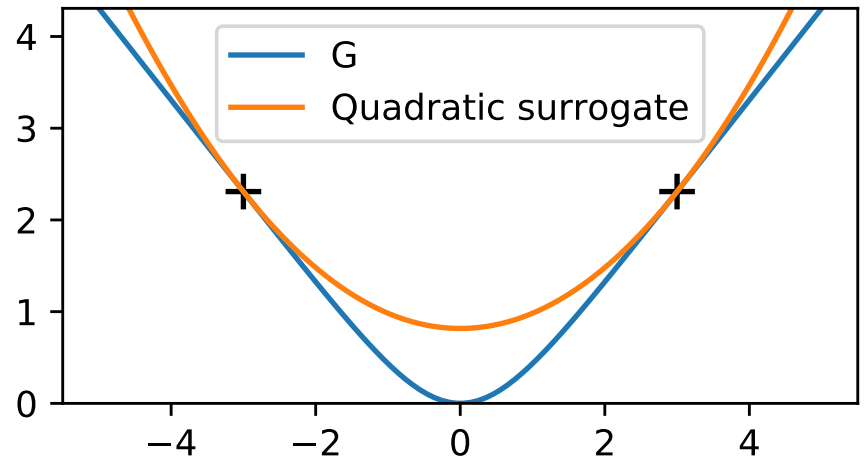
$G(\sqrt{\cdot})$ is concave.

- This is the case for most brain sources



Main idea: surrogate functions

G has a quadratic surrogate at each point.



$$G(y) = \min_{u \geq 0} \frac{uy^2}{2} + f(u)$$

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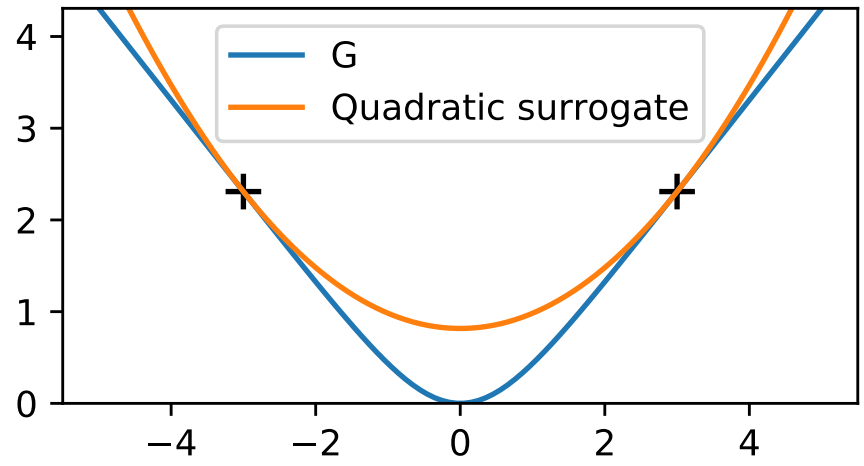

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- minimum reached for an unique value $u^*(y) = \frac{G'(y)}{y}$.

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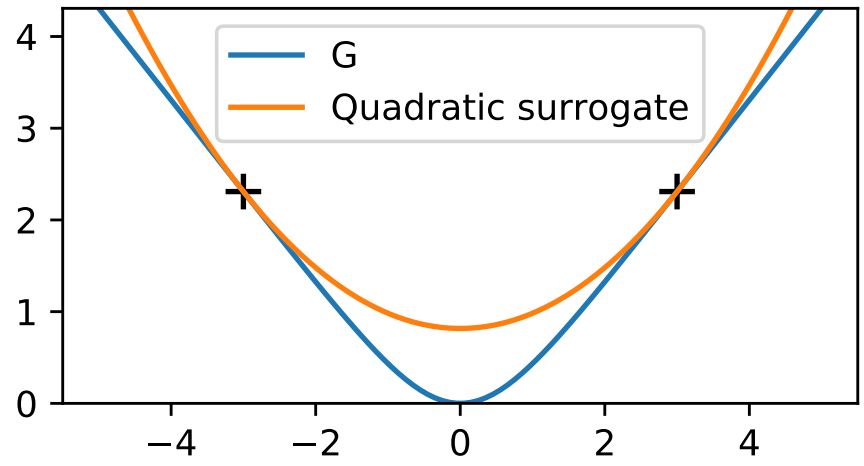

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Surrogate risk

$$G(y) = \min_{u \geq 0} \frac{uy^2}{2} + f(u)$$

$$\ell(\mathbf{x}, W) = -\log|\det(W)| + \sum_{i=1}^{p} G([W\mathbf{x}]_i)$$

Introduce dual variables $\mathbf{u} \in \mathbb{R}^{p \times 1}$:

$$\tilde{\ell}(\mathbf{x}, W, \mathbf{u}) = -\log|\det(W)| + \frac{1}{2}\sum_{i=1}^{p} u_i \, [W\mathbf{x}]_i^2 + \sum_{i=1}^{p} f(u_i)$$

- Much simpler dependence in W !

Surrogate loss

$$G(y) = \min_{u \geq 0} \frac{uy^2}{2} + f(u)$$

$$\mathcal{L}_n(W) = -\log|\det(W)| + \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n} G([WX]_{ij})$$

Introduce dual variables $U \in \mathbb{R}^{p \times n}$:

$$\tilde{\mathcal{L}}_n(W, U) = -\log|\det(W)| + \frac{1}{2n}\sum_{i=1}^{p}\sum_{j=1}^{n} U_{ij}\,[WX]_{ij}^2 + \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n} f(U_{ij})$$

Majorization properties

$$\mathcal{L}_n(W) = -\log|\det(W)| + \frac{1}{n}\sum_{i=1}^{p}\sum_{j=1}^{n} G([WX]_{ij})$$

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- $\mathcal{L}_n(W) \leq \tilde{\mathcal{L}}_n(W,U)$, with equality iif $U = u^*(WX)$
- W minimizes $\mathcal{L}_n$ if and only if $(W, u^*(WX))$ minimizes $\tilde{\mathcal{L}}_n$.

Alternate minimization

Idea:

- For a fixed U, minimize $\tilde{\mathcal{L}}_n(W, U)$ w.r.t. W
- For a fixed W, minimize $\tilde{\mathcal{L}}_n(W, U)$ w.r.t. U

Minimization in W

$$\tilde{\mathcal{L}}_n(W, U) = -\log|\det(W)| + \frac{1}{2n}\sum_{i=1}^{p}\sum_{j=1}^{n} U_{ij}\ [WX]_{ij}^2 + \cdots$$

Quadratic function in the rows of W:

$$\tilde{\mathcal{L}}_n(W, U) = -\log|\det(W)| + \frac{1}{2}\sum_{i=1}^{p} W_{i:}A^i W_{i:}^\top + \cdots$$

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$$A_{kl}^i = \frac{1}{n}\sum_{j=1}^{n} U_{ij} X_{kj} X_{lj}$$

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Minimization possible w.r.t. a *multiplicative* update of $W_{i:}$: $W \leftarrow MW$ where M is identity except for its i-th row which equals $\mathbf{m} \in \mathbb{R}^p$.

W.r.t $\mathbf{m}$, $\tilde{\mathcal{L}}_n(MW,U)$ is of the form $-\log(|m_i|) + \frac{1}{2}\mathbf{m}K\mathbf{m}^\top$, $K = WA^iW^\top \in \mathbb{R}^{p\times p}$.

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Finite sum setting: n fixed, minimize $\tilde{\mathcal{L}}_n$.

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Online algorithm

Streaming setting: you recieve samples one at a time. You can only use a sample once. n is not fixed.

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All good!

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Results

Convergence measures

- **Loss on left-out data**
- **Amari distance** Requires that the true mixing matrix A is available. For a matrix W, compute $R = WA$ and
  $d = \sum_{i=1}^{p}(\sum_{j=1}^{p} \frac{R_{ij}^2}{\max_l R_{il}^2} - 1) + \sum_{i=1}^{p}(\sum_{j=1}^{p} \frac{R_{ji}^2}{\max_l R_{lj}^2} - 1)$.
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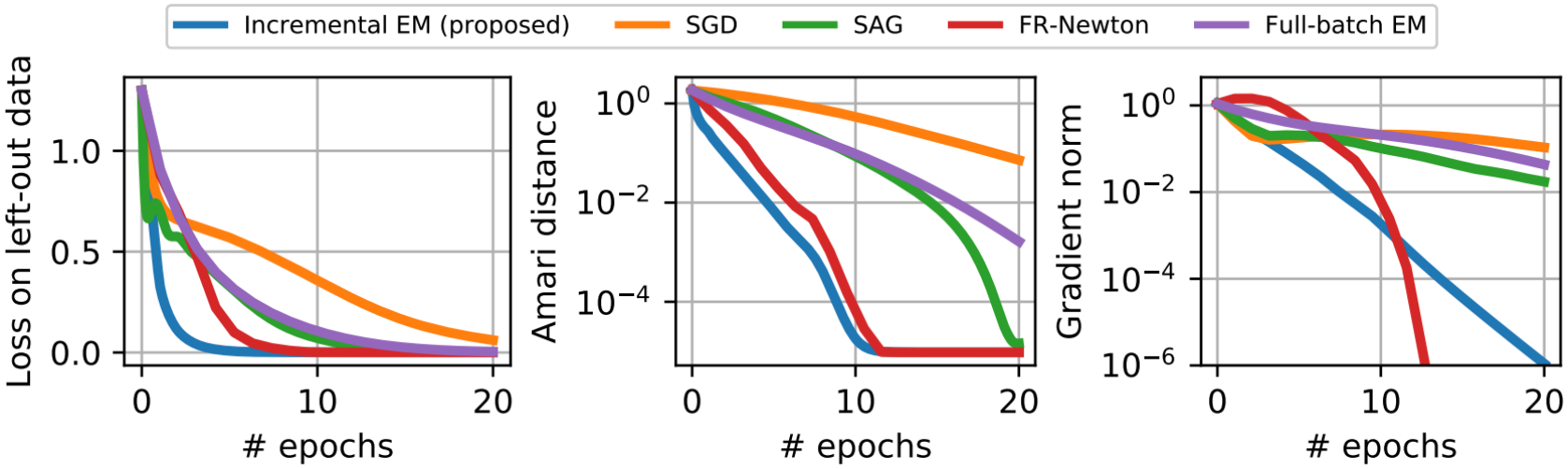
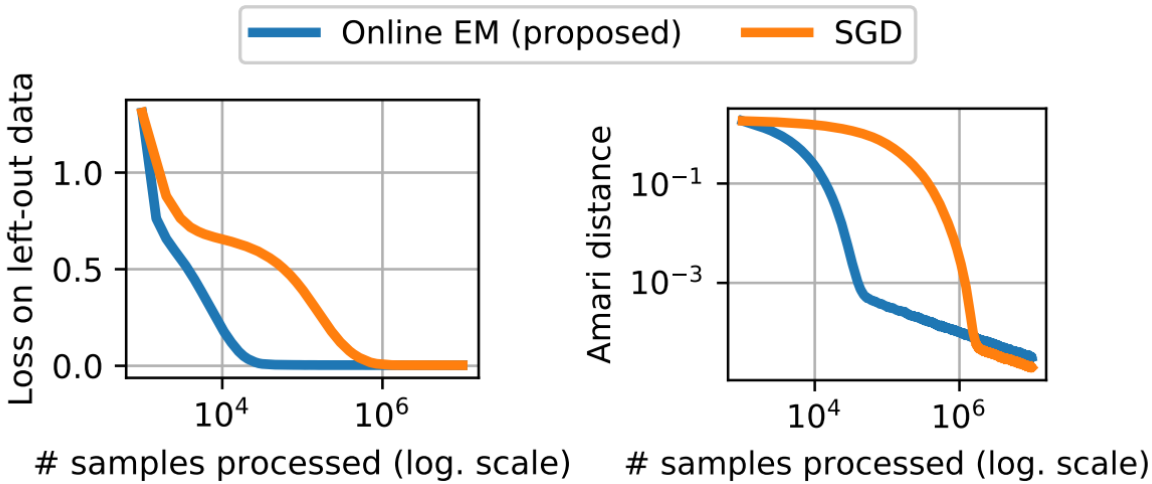
Simulated data

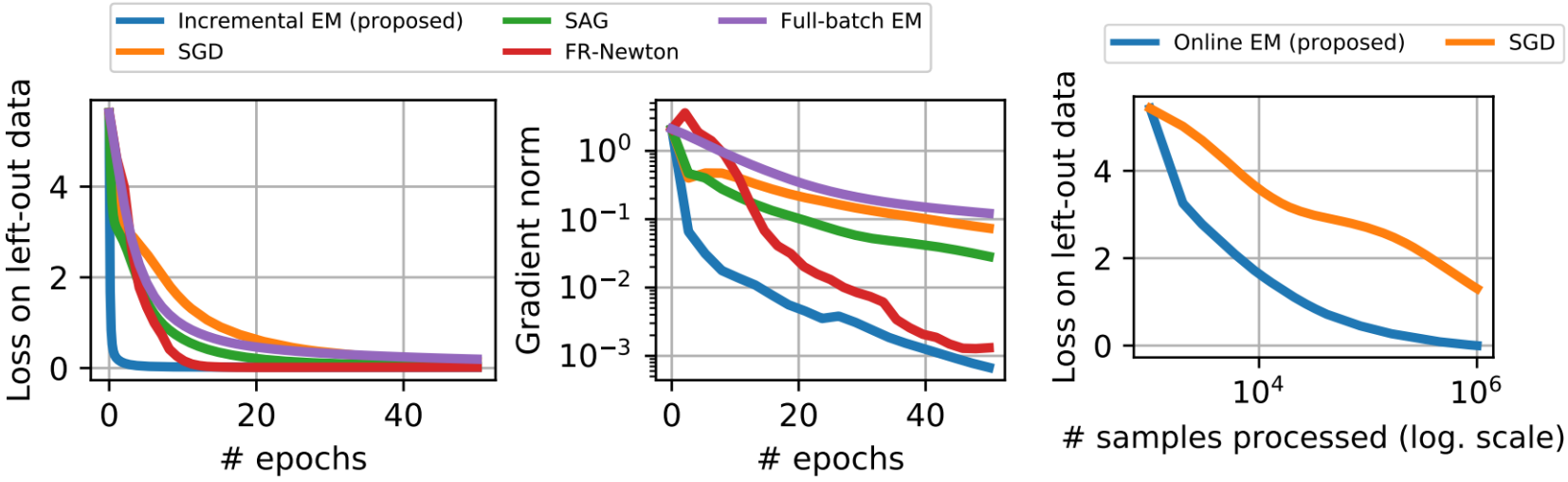
$p = 10$, $n = 10^6$ in the finite sum setting, 10^7 in the online setting.
$S \in \mathbb{R}^{p \times n}$ generated with density $d(x) = \frac{1}{2}\exp(-|x|)$. $X = AS$





EEG data

$$p = 30,\ n = 10^6$$



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Thanks for your attention!